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The Use of the Galerkin Method with a Basis of B-Splines for the Solution of the One-Dimensional Primitive Equations

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Numerical solutions of the primitive equations are obtained for a number of boundary conditions, using the Galerkin method with a basis of *B*-splines. The splines are generated by a recurrence formula that allows for multiple knots, and advantage is taken of this in incorporating boundary conditions. The necessary integrals that occur are readily evaluated by expanding the *B*-splines in terms of Chebyshev polynomials, and the effect upon accuracy of changes in the basis set and order of the spline is considered.

1. INTRODUCTION

In recent years semianalytic techniques have emerged as strong competitors to finite difference methods for the solution of multidimensional partial differential equations. In the semianalytic methods one or more dependent variables are expanded as a series of functions and associated coefficients. There are a number of ways in which these coefficients can be determined, however, the Galerkin [1] method is one commonly employed.

These techniques have been used extensively, in the form of the finite element method, in solving the partial differential equations arising in structural mechanics, but it is only lately that they have been applied to solving time dependent problems arising in meteorology and oceanography.

Orszag [2, 3] has used the spectral method, in which a solution is obtained in terms of functions which are continuous over the whole spatial domain (usually Chebyshev polynomials, Legendre functions, or a Fourier expansion) to solve a number of meteorological problems.

The finite element method, in which the basis consists of low-order piecewise polynomials, has been applied by Cullen [4] to the integration of the shallow water equations on a sphere. The advantages of the finite element method, in terms of computer storage, time, and accuracy have been demonstrated by Wang *et al.* [5].

In oceanography Rao [6] proposes a type of spectral method for the calculation of storm surges in a lake, using the normal modes of the lake as his basis functions. The finite element method has been applied by Grotkop [7] to the calculation of the M_2 tidal constituent in the North Sea.

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In the present paper the Galerkin method is used, with a basis of B-splines, on a number of one-dimensional problems that arise in meteorology and oceanography for which analytical solutions are available. It will be shown that a set of B-splines can produce very accurate solutions, even when only a small basis set is employed, and that the boundary conditions imposed by the problem can readily be satisfied by the basis. The splines are generated by a simple recursion formula, which allows for multiple knots, and any order of spline. By expanding these splines in terms of Chebyshev polynomials, all the necessary integrals can be evaluated by recurrence relationships.

2. Model Equations and the Galerkin Approximation

In one dimension the equations of motion and continuity for a single homogeneous layer of fluid, neglecting friction and advective terms, are given by:

$$\frac{\partial u}{\partial t}(x, t) + g\frac{\partial \xi}{\partial x}(x, t) = 0 \tag{1}$$

and

$$\frac{\partial \xi}{\partial t}(x, t) + H\frac{\partial u}{\partial x}(x, t) = 0 \tag{2}$$

where H, a constant, is the mean depth of fluid, g is the acceleration due to gravity, u is the fluid speed in the x direction, and ξ is the total depth of the fluid. In order to obtain unique solutions for (1) and (2), initial conditions $u(x, 0)$ and $\xi(x, 0)$ at time $t = 0$ and boundary conditions must be specified. The form of these is considered later, when the method is applied to a number of problems.

Expressing $u(x, t)$ and $\xi(x, t)$ as an expansion of B-spline functions $M_{ni}(x)$ and time dependent coefficients $\alpha_i(t)$ and $\beta_i(t)$ gives:

$$u(x, t) = \sum_{i=1}^{p} \alpha_i(t)\, M_{ni}(x) \tag{3}$$

and

$$\xi(x, t) = \sum_{i=1}^{p} \beta_i(t)\, M_{ni}(x) \tag{4}$$

where n is the order of the spline and p is the number of spline functions in the expansion.

Substituting Eqs. (3) and (4) into (1) and (2), and using the Galerkin method (solutions for u and ξ being required in the region $0 \leqslant x \leqslant L$) gives

$$\sum_{i=1}^{p} \frac{d\alpha_i}{dt} \int_0^L M_{ni} M_{nj}\, dx + g \sum_{i=1}^{p} \beta_i \int_0^L \frac{dM_{ni}}{dx} M_{nj}\, dx = 0 \tag{5}$$

and

$$\sum_{i=1}^{p} \frac{d\beta_i}{dt} \int_0^L M_{ni} M_{nj}\, dx + H \sum_{i=1}^{p} \alpha_i \int_0^L \frac{dM_{ni}}{dx} M_{nj}\, dx = 0 \tag{6}$$

where $j = 1, 2, \ldots, p$.

Swartz and Wendroff [8] have shown that the Galerkin method can be applied to problems of the general form,

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbb{A}\mathbf{u} \tag{7}$$

yielding error estimates $O(h^{n-1})$ where n is the order of spline, and h is the grid interval of a uniform grid. Equations (1) and (2) constitute a particular form of Eq. (7) with

$$\mathbf{u} = \begin{pmatrix} u(x, t) \\ \xi(x, t) \end{pmatrix}, \qquad \mathbb{A} = \begin{bmatrix} 0, & -g \dfrac{\partial}{\partial x} \\ -H \dfrac{\partial}{\partial x}, & 0 \end{bmatrix}. \tag{8}$$

The set of coupled differential equations (5) and (6) can be integrated through time using any standard numerical method. A fourth order Runge–Kutta technique was used here to integrate these equations. Initial values $\alpha_i(0)$ and $\beta_i(0)$ can be calculated by expanding the initial conditions $u(x, 0)$ and $\xi(x, 0)$ in terms of *B*-splines and integrating over the spatial domain, yielding:

$$\sum_{i=1}^{p} \alpha_i(0) \int_0^L M_{ni} M_{nj}\, dx = \int_0^L u(x, 0)\, M_{nj}\, dx \tag{9}$$

and

$$\sum_{i=1}^{p} \beta_i(0) \int_0^L M_{ni} M_{nj}\, dx = \int_0^L \xi(x, 0)\, M_{nj}\, dx \tag{10}$$

where $j = 1, 2, \ldots, p$.

Integrals involving only spline functions can be evaluated analytically. For well-behaved $u(x, 0)$, $\xi(x, 0)$ integrals involving these functions were approximated using Gauss quadrature.

3. Form of the Basis Functions and Numerical Procedure

Solution of Eqs. (5) and (6) requires the evaluation of integrals over the basis functions. The ease with which values of these are obtained depends upon the nature of the basis set. Spline functions can be readily integrated and differentiated because they are piecewise polynomials, and since they have a basis with small support, many of the integrals that occur are zero when this basis is used.

The basis functions used here are the B-splines (fundamental splines), which were first studied by Schoenberg [9] and have recently been used extensively in problems of interpolating and smoothing by Powell [10] and Schumaker [11]. Using an increasing set of N interior knots, and n knots for support, positioned outside the region $0 \leqslant x \leqslant L$, at

$$\lambda_{1-n} \leqslant \lambda_{2-n} \leqslant \cdots \leqslant \lambda_0 = 0$$

and

$$L = \lambda_{N+1} \leqslant \lambda_{N+2} \leqslant \cdots \leqslant \lambda_{N+n}$$

the B-spline functions can readily be evaluated by the numerically stable method given by Cox [12] and de Boor [13, 14]. This method permits an arbitrary knot spacing and the knots may coincide.

In the solution of the various problems described later, a constant interior knot spacing was used. Thus h the grid distance (the inverse of the number of grid lengths), is the distance between adjacent interior knots divided by L. The grid distance being reduced by increasing the number of interior knots, corresponding to an increase in basis size.

Rather than evaluating the integrals involving the spline functions directly, it is convenient to express the splines in terms of Chebyshev polynomials. Over each knot interval $\lambda_j \leqslant x \leqslant \lambda_{j+1}$ the B-spline is a polynomial of degree $n-1$ or less, and can be expressed as

$$M_{nj}(x) = \sum_{i=0}^{n-1}{}'' \gamma_{ji} T_i(X) \tag{11}$$

where $X = (2x - \lambda_j - \lambda_{j+1})/(\lambda_{j+1} - \lambda_j)$, for $-1 \leqslant X \leqslant 1$, and $T_i(X)$ is a Chebyshev polynomial of the first kind. The double prime indicates that the first and last γ is to be halved when the sum is evaluated. The γ_{ji} is given by (Fox and Parker [15])

$$\gamma_{ji} = \frac{2}{n-1} \sum_{k=0}^{n-1}{}'' T_i(X_k)\, M_j(x_k) \qquad (i = 0, 1, \ldots, n-1) \tag{12}$$

with

$$X_k = \cos\frac{k\pi}{n-1} \qquad \text{and} \qquad x_k = \frac{X_k(\lambda_{j+1} - \lambda_j) + \lambda_j + \lambda_{j+1}}{2}.$$

Using the transformation given in (11), the integrals involved in Eqs. (5) and (6) can be expressed in terms of integrals of Chebyshev polynomials

$$\frac{\lambda_{j+1} - \lambda_j}{2} \int_{-1}^{+1} T_r(X)\, T_s(X)\, dX \qquad \text{and} \qquad \int_{-1}^{+1} T_r(X) \frac{dT_s}{dX}(X)\, dX.$$

Making the substitution $X = \cos\theta$ and using:

$$T_r(X)\,T_s(X) = 0.5(T_{s+r}(X) + T_{|s-r|}(X)), \tag{13}$$

$$T_r(X) = \cos(r \cos^{-1} X) \tag{14}$$

these integrals can be readily evaluated.

4. Application of the Method to Specific Problems

(a) *Periodic Boundary Conditions*

The numerical example used by Wang *et al.* [5] provides a good test of the application of the method to a problem having periodic boundary conditions.

The initial conditions are:

$$u(x, 0) = U_0 \sin 2\pi r x/L$$

and

$$\xi(x, 0) = H. \tag{15}$$

Boundary conditions are given by:

$$u(x, t) = u(x + L, t),$$
$$\xi(x, t) = \xi(x + L, t), \tag{16}$$

where U_0 is a constant amplitude, r is an integer determining the wavelength, H, a constant, is the depth of fluid, taken as 9.184 km, and $L = 10{,}500$ km, $g = 9.81$ m/sec^2.

To test the accuracy of the method, $\xi(x, t)$ and $u(x, t)$ were calculated at 200 equally spaced points in the region $0 \leqslant x \leqslant L$ at each time step, and the maximum differences between these and the analytic solution given by Wang *et al.* [5] were obtained. The maximum value of the error in $\xi(\Delta\xi)$ and $u(\Delta u)$ (normalized by dividing by H and U_0, respectively) which occurred during each integration period is given in the tables.

Table I presents the results for $r = 1.0$, $U_0 = 54.6$ m/sec, using a cubic B-spline ($n = 4$) in both single precision (s.p.) (7-figure accuracy) and double precision (d.p.) (16-figure accuracy). A time step of 600 sec (approximately 1/60 of the period) was used, except in the case of 18 grid lengths where a smaller time step (300 sec) was required to prevent numerical errors in the time integration having a large effect.

Comparing single-precision and double-precision results in Table I at 10 hr in particular, it is evident that as the number of grid lengths increases (i.e., h decreases), precision problems associated with the inversion of the matrix of integrals having i, j element $\int_0^L M_i M_j \, dx$ (used in solving Eqs. (5) and (6) with the Runge–Kutta method) and the solution of the resulting set of coupled differential equations become more important, as indicated by the deterioration of the single-precision results.

TABLE I

Errors Using Fourth Order *B*-Splines ($r = 1.0$), with a Periodic Boundary Condition

Time (hr)		6 grid lengths		12 grid lengths		18 grid lengths	
		s.p.	d.p.	s.p.	d.p.	s.p.	d.p.
10	$\Delta\xi$	3.97*E*–4[a]	3.90*D*–4[b]	3.31*E*–5	2.01*D*–5	4.68E–5	4.13*D*–6
	ΔU	2.16*E*–3	1.96*D*–3	1.65*E*–4	1.85*D*–4	1.17*E*–4	9.38*D*–5
20	$\Delta\xi$	4.65*E*–4	3.91*D*–4	4.69*E*–5	2.01*D*–5	8.86*E*–5	5.35*D*–6
	ΔU	2.40*E*–3	1.98*D*–3	2.25*E*–4	2.05*D*–4	2.23*E*–4	1.69*D*–4
30	$\Delta\xi$	5.51*E*–4	3.92*D*–4	6.38*E*–5	2.00*D*–5	1.35*E*–4	7.70*D*–6
	ΔU	2.66*E*–3	2.01*D*–3	2.88*E*–4	2.53*D*–4	3.55*E*–4	2.51*D*–4
40	$\Delta\xi$	6.37*E*–4	3.95*D*–4	8.73*E*–5	2.01*D*–5	1.81*E*–4	1.01*D*–5
	ΔU	2.91*E*–3	2.05*D*–3	3.48*E*–4	3.02*D*–4	4.60*E*–4	3.37*D*–4
50	$\Delta\xi$	7.21*E*–4	3.99*D*–4	1.03*E*–4	2.01*D*–5	2.29*E*–4	1.25*D*–5
	ΔU	3.17*E*–3	2.11*D*–3	4.21*E*–4	3.50*D*–4	5.75*E*–4	4.27*D*–4
60	$\Delta\xi$	8.40*E*–4	4.02*D*–4	1.20*E*–4	2.02*D*–5	2.75*E*–4	1.49*D*–5
	ΔU	3.42*E*–3	2.16*D*–3	4.91*E*–4	3.99*D*–4	7.07*E*–4	5.18*D*–4

[a] 3.97*E*–4 = 3.97×10^{-4} *E* indicating single precision.
[b] 3.90*D*–4 = 3.90×10^{-4} *D* indicating double precision.

Dupont [16] gives error bounds, C_1h^4 and C_2h^3, C_1 and C_2 positive constants, when the Galerkin method is applied to the solution of this problem using a basis of smooth cubic splines and Hermite cubic functions, respectively. From these error estimates it is evident that errors of a similar order of magnitude arise using cubic splines with 6 grid lengths, as with 12 Hermite cubic functions. Wang *et al.* [5] solve Eqs. (1) and (2) using initial conditions (15) and boundary conditions (16) with Hermite cubics and obtain after a 10-hr integration period, with a basis of 12 functions, $\Delta\xi = 2.13D\text{–}4$ compared with $\Delta\xi = 3.90D\text{–}4$ (Table I, cubic splines, 6 grid lengths).

Errors for $r = 2.0$, $U_0 = 27.3$ m/sec should be of the same order of magnitude with a basis of 18 Hermite cubics, as cubic splines with 10 grid lengths. Wang *et al.* [5] using a basis of 18 Hermite cubic functions with a time step of 1.25 min give $\Delta\xi$ (10 hr) = 2.79*D*–4 compared with 4.55*D*–4 (Table III, cubic splines, 10 grid lengths). Although a different order of time integration, and size of time step were used here from that of Wang *et al.* [5], comparison of the computed errors appears in reasonable agreement with the theoretical results of Dupont [16].

Since comparable accuracy is obtained using cubic splines, 10 grid lengths, a time step of 10 min, and the fourth-order Runge–Kutta technique to that obtained with 18 Hermite cubic functions and a smaller time step, the use of cubic splines with the Runge–Kutta method appears computationally more economical.

It is evident from the error bounds for cubic splines that having h should diminish the error by a factor of 16. Comparing double-precision errors in Table I for 6 and 12 grid lengths, the error in elevation is reduced by a factor of 19, and the error is current by a factor of 11, in reasonable agreement with theoretical estimates, although as the number of grid lengths increases the reduction in error diminishes due presumably to increased rounding error.

Swartz and Wendroff [17, 18] examine the errors which arise in applying the finite element method with a basis of smooth spline functions to the model problem,

$$\frac{\partial u}{\partial t} = c \frac{\partial u}{\partial x}. \tag{17}$$

They derive an error estimate E for the solution of (17), with sinusoidal initial conditions and periodic boundary conditions, of the form,

$$E = -2\pi N \frac{\sum_{j=-\infty}^{\infty} \frac{j}{(1/N + j)^{2n}}}{\sum_{j=-\infty}^{\infty} \frac{1}{(1/N + j)^{2n}}} \tag{18}$$

where N = number of grid intervals per wavelength, and n is as defined previously.

TABLE II

Errors Using Sixth-Order *B*-Splines ($r = 1.0$) with a Periodic Boundary Condition

Time (hr)		6 grid lengths s.p.	6 grid lengths d.p.	12 grid lengths s.p.	12 grid lengths d.p.
10	$\Delta\xi$	1.07*E*–4	1.35*D*–5	2.03*E*–5	3.45*D*–6
	ΔU	1.57*E*–4	1.16*D*–4	1.74*E*–4	9.29*D*–5
20	$\Delta\xi$	1.82*E*–4	1.35*D*–5	3.57*E*–5	6.47*D*–6
	ΔU	2.62*E*–4	1.60*D*–4	3.16*E*–4	1.91*D*–4
30	$\Delta\xi$	2.61*E*–4	1.35*D*–5	5.03*E*–5	9.49*D*–6
	ΔU	3.72*E*–4	2.05*D*–4	4.85*E*–4	2.95*D*–4
40	$\Delta\xi$	3.40*E*–4	1.42*D*–5	6.71*E*–5	1.25*D*–5
	ΔU	4.73*E*–4	2.53*D*–4	6.50*E*–4	4.05*D*–4
50	$\Delta\xi$	4.19*E*–4	1.56*D*–5	8.35*E*–5	1.56*D*–5
	ΔU	5.59*E*–4	3.02*D*–4	8.19*E*–4	5.16*D*–4
60	$\Delta\xi$	4.91*E*–4	1.67*D*–5	1.00*E*–4	1.86*D*–5
	ΔU	6.48*E*–4	3.52*D*–4	9.96*E*–4	6.30*D*–4

TABLE III

Errors Using Fourth- and Sixth-Order B-Splines ($r = 2.0$) with a Periodic Boundary Condition

Time (hr)		Fourth-order splines, number of grid lengths 6	10	12	Sixth-order splines, number of grid lengths 6	10	12
10	$\Delta\xi$	1.46D–2	4.55D–4	1.93D–4	1.63D–3	3.14D–5	2.10D–5
	ΔU	6.57D–2	2.42D–3	1.04D–3	8.55D–3	2.45D–4	1.94D–4
20	$\Delta\xi$	2.78D–2	4.83D–4	1.99D–4	2.28D–3	5.23D–5	4.04D–5
	ΔU	1.38D–1	2.64D–3	1.17D–3	1.09D–2	3.97D–4	3.88D–4
30	$\Delta\xi$	4.13D–2	5.31D–4	2.11D–4	2.95D–3	7.09D–5	6.03D–5
	ΔU	2.11D–1	2.95D–3	1.31D–3	1.46D–2	5.64D–4	5.84D–4
40	$\Delta\xi$	5.46D–2	5.98D–4	2.29D–4	3.72D–3	9.04D–5	8.08D–5
	ΔU	2.80D–1	3.33D–3	1.49D–3	1.96D–2	7.45D–4	7.95D–4
50	$\Delta\xi$	6.77D–2	6.76D–4	2.50D–4	4.52D–3	1.10D–4	1.01D–4
	ΔU	3.66D–1	3.75D–3	1.68D–3	2.42D–2	9.73D–4	1.03D–3
60	$\Delta\xi$	8.01D–2	7.59D–4	2.75D–4	5.35D–3	1.30D–4	1.21D–4
	ΔU	4.36D–1	4.29D–3	1.92D–3	2.91D–2	1.15D–3	1.23D–3

It is interesting to compare errors derived using (18) for this model problem with those calculated numerically at $t = 10$ hr (Tables I–III), for Eqs. (1) and (2) with initial conditions (15) and periodic boundary conditions (16).

For fourth-order splines ($n = 4$) and six grid intervals per wavelength ($N = 6$) Eq. (18) gives $E = 9.02D$–5 compared with $\Delta\xi = 3.90D$–4 (Table I, 6 grid lengths) and for $n = 4$, $N = 12$, $E = 2.60D$–7 compared with $\Delta\xi = 2.01D$–5 (Table I, 12 grid lengths). With sixth-order splines ($n = 6$, $N = 6$) Eq. (18) gives $E = 1.52D$–7 and for $n = 6$, $N = 12$, $E = 2.08D$–11 compared with $\Delta\xi = 1.35D$–5 and 3.45D–6, respectively (Table II).

From these results it is evident that the numerical solutions are not as accurate as error estimates derived from the model equation suggest. It is obvious comparing single-precision and double-precision results in Tables I and II that rounding errors reduce the accuracy of the numerical solution as the number of terms in the expansion and order of spline increase and hence for the higher N values considered here the numerical result will contain a larger error than that suggested by theory. However, for the case $r = 2.0$, $n = 4$, with six grid intervals (i.e. $N = 3$) the numerical result $\Delta\xi = 1.46D$–2, $\Delta u = 6.57D$–2 (Table III) compares well with the error estimate $E = 7.31D$–2, and for sixth order splines ($n = 6$, $N = 3$) the numerical result $\Delta\xi =$ 1.63D–3, $\Delta u = 8.55D$–3 is also comparable with the error estimate $E = 4.60D$–3. These results illustrate the increase in accuracy obtained using the higher order spline.

In these two cases the computed error is fairly large and the effect of rounding is presumably small in comparison.

From the error estimate (Eq. (18)) it is evident that for a given n and N, the error is independent of the number of grid intervals. Comparing numerical results at $t =$ 10 hr, $n = 6$, $r = 2.0$, $\Delta\xi = 2.10D\text{–}5$ using 12 grid intervals with $\Delta\xi = 1.35D\text{–}5$ for $r = 1.0$, $n = 6$; using 6 grid intervals it is evident that comparable errors are also obtained numerically for a given n, N, with a different number of grid intervals, although from Tables I–III it is evident that as the number of grid intervals increases for a given n, N the error also increases.

Although these calculations were performed for equally spaced knots, the method for evaluating the B-splines allows for unequal knot intervals.

(b) *Specification of Elevation and Current at the Same Boundary*

Periodic boundary conditions as defined in the preceding problem are rarely encountered in oceanography.

An example of more realistic character is concerned with a progressive wave in a channel, where the elevation and current on the boundary $x = 0$ vary in a periodic manner, given by

$$\begin{aligned} \xi(0, t) &= C \cos \omega t = z_0(t), \\ u(0, t) &= AC \cos \omega t = u_0(t). \end{aligned} \tag{19}$$

The analytic solutions of (1) and (2) subject to these boundary conditions are given by

$$\begin{aligned} \xi(x, t) &= C \cos (rx - \omega t), \\ u(x, t) &= AC \cos(rx - \omega t), \end{aligned} \tag{20}$$

where $A = (g/H)^{1/2}$, r determines the wavelength, C is the amplitude, and $\omega = r(gH)^{1/2}$ is the period. The solution represents a progressive wave traveling along the channel from $x = 0$ to $x = L$.

The implementation of boundary conditions can be accomplished by two methods. If the knot at $x = 0$ has a multiplicity of $n - 1$, then all the B-splines in expansions (3) and (4) except the first will vanish at this point. Thus the expansions for ξ and u which satisfy the boundary conditions become

$$u(x, t) = \frac{u_0(t)}{A_1} M_1(x) + \sum_{i=2}^{p} \alpha_i M_i(x) \tag{21}$$

and

$$\xi(x, t) = \frac{z_0(t)}{A_1} M_1(x) + \sum_{i=2}^{p} \beta_i M_i(x) \tag{22}$$

where $A_1 = M_1(0)$, and p is the number of basis functions.

The unknown coefficients α_i and β_i are determined as before, using $M_2, M_3, \ldots, M_p$ as test functions giving two sets of $(p - 1)$ equations.

In the second method, a single knot is placed at $x = 0$. In this case the first $n - 1$ B-splines will be nonzero at this point, and in order to satisfy the imposed boundary conditions the differential equations given by (5) and (6) must be solved subject to the constraints imposed by the boundary conditions on the first $n - 1$ α's and $n - 1$ β's, namely:

$$\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_{n-1} A_{n-1} = u_0(t) \tag{23}$$

and

$$\beta_1 A_1 + \beta_2 A_2 + \cdots + \beta_{n-1} A_{n-1} = z_0(t) \tag{24}$$

where $A_i = M_i(0)$.

In the previous problem, where periodic boundary conditions were used, these conditions imposed the constraint that the wave had the same amplitude and phase at $x = 0$ and $x = L$. In this problem that constraint is no longer required, and values of rL which are not integer multiples of π were chosen to ensure different amplitudes and phases at $x = 0$ and $x = L$, namely, (i) $rL = 1.6\pi$, (ii) 3.2π, (iii) 6.4π. The value of rL is important since from (20) it determines the number of wavelengths present.

TABLE IV

Variation of Errors with Time for Problem (b) (ii) Using Fourth-Order B-Splines

Time cycles		Number of grid lengths 5	10	15
3	$\Delta\xi$	4.92*D*–1	4.66*D*–3	5.85*D*–4
6	$\Delta\xi$	5.23*D*–1	4.63*D*–3	5.61*D*–4
9	$\Delta\xi$	4.72*D*–1	4.27*D*–3	5.71*D*–4
12	$\Delta\xi$	4.92*D*–1	4.80*D*–3	5.66*D*–4
15	$\Delta\xi$	5.21*D*–1	4.60*D*–3	5.62*D*–4

Initial conditions corresponding to (19) with $t = 0$ were used, together with a time step of 1/60 of the period. After a few cycles, the error reached a value determined by the periodic forcing of the boundary conditions, and the rounding error in the calculation. Table IV gives the normalized maximum error found at 200 points in the elevation for case (ii) $rL = 3.2\pi$. The current is related to the elevation by the constant A. Table IV shows that the error is fairly constant after the first few cycles.

Identical calculations were performed for the three waves, cases (i), (ii), and (iii), with both fourth- and sixth-order B-splines; the results after 15 cycles are summarized in Table V.

TABLE V

Errors after 15 Cycles for Problem (b), Using a Number of Wavelengths (Cases (i), (ii), and (iii))

		Fourth-order splines, number of grid lengths			Sixth-order splines, number of grid lengths		
Case		5	10	15	5	10	15
(i)	$\Delta\xi$	3.91D–3	1.59D–4	2.75D–5	1.69D–4	1.01D–5	1.64D–5
(ii)	$\Delta\xi$	5.21D–1	4.60D–3	5.62D–4	5.69D–2	2.30D–4	2.11D–5
(iii)	$\Delta\xi$	1.27D–0	2.56D–1	2.98D–2	1.06D–0	3.62D–2	1.82D–3

Calculations for all cases were also repeated using the method of constraining the expansion coefficients to satisfy the boundary conditions; and gave results which agreed with Table V.

Swartz and Wendroff [8] give error estimates $O(h^{n-1})$ for this problem, hence for fourth-order B-splines, halving the grid spacing reduces the error by a factor of 8, and for sixth-order B-splines by a factor of 32.

Comparing errors (Table V, case (i)) for 5 and 10 grid lengths, for fourth-order B-splines the error is reduced by a factor of 25, and for sixth-order B-splines by a factor of 16, although for case (iii) the error is reduced by a factor of 5 and 30, respectively, in reasonable agreement with theory. In case (i), sixth-order B-splines, five grid lengths, the error is fairly small, and it is obviously affected appreciably by rounding error as the number of basis functions is increased, actually becoming larger when the number of grid lengths is increased from 10 to 15.

Comparing errors from cases (i) and (ii) (Table V), the effect of halving the wavelength can be determined, and it is obvious that to maintain the same order of accuracy the number of grid lengths must double, halving h; e.g., for fourth-order splines (case (i)) $\Delta\xi = 3.91D\text{–}3$ with 5 grid lengths, compared with case (ii),

$\Delta\xi = 4.60D\text{–}3$ with 10 grid lengths, and a similar result occurs for sixth-order B-splines.

From the error estimate it is obvious that the error is reduced as the order of spline increases, although from Table V it is evident that this improvement diminishes from case (i) to case (iii). Thus with 5 grid lengths, using a sixth-order B-spline instead of a fourth-order B-spline, for case (i) the error is reduced by a factor of 23, for case (ii) by 9, and for case (iii) by 1.2.

(c) *Elevation and Current Specified at Opposite Boundaries*

Consider a gulf connected to a tidal ocean, with the closed end of the gulf at $x = 0$ and the open end at $x = L$. At $x = L$ there is a forced sinusoidal variation in the elevation.

The appropriate boundary conditions are

$$u(0, t) = 0 \qquad \text{for all } t, \qquad \xi(L, t) = A \cos \omega t. \tag{25}$$

Analytic solutions of (1) and (2) with these boundary conditions give

$$u(x, t) = AC \frac{\sin rx}{H \cos rL} \sin \omega t, \qquad \xi(x, t) = A \frac{\cos rx}{\cos rL} \cos \omega t, \tag{26}$$

where A is the amplitude of the forcing sinusoid, ω is the frequency, $C = (gH)^{1/2}$, and $r = \omega/C$ determines the wavelength. Only the value of rL is important in determining the number of basis functions required; however, since the problem is one that occurs frequently in oceanography, typical values of the other parameters were used, namely, $L = 300$ km, $H = 90.8$ m, $g = 9.81$ m/sec^2, $A = 1.0$ m.

Two calculations were performed using (i) $rL = 6.0$ and (ii) $rL = 12.0$. Initial conditions corresponding to (26) with $t = 0$ were used, together with a time step 1/60 of a period. After a few cycles the error reached a nearly constant value, and the maximum errors $\Delta\xi$ and Δu after 15 cycles are given in Table VI.

TABLE VI

Errors after 15 Cycles for Problem (c), Cases (i) and (ii)

Case		Fourth-order *B*-splines, number of grid lengths			Sixth-order *B*-splines, number of grid lengths		
		6	10	12	6	10	12
(i)	$\Delta\xi$	7.09*D*–3	4.32*D*–4	1.76*D*–4	2.98*D*–4	2.66*D*–4	1.57*D*–4
	ΔU	3.47*D*–3	1.99*D*–4	7.64*D*–5	1.07*D*–4	1.02*D*–4	5.70*D*–5
(ii)	$\Delta\xi$	1.10*D*–0	7.74*D*–2	2.29*D*–2	3.01*D*–1	9.49*D*–4	4.79*D*–4
	ΔU	2.96*D*–1	2.63*D*–2	8.03*D*–3	1.04*D*–2	3.94*D*–4	1.99*D*–4

The boundary conditions were satisfied by applying constraints to the expansion coefficients as described previously. This example is of interest not only because boundary conditions are applied to ξ and u at opposite ends of the channel, but unlike the case of the progressive wave, the elevation and current are out of phase in both space and time. From Table VI it is evident that doubling the number of grid lengths for fourth-order *B*-splines in both cases (i) and (ii) reduces the error by a factor of approximately 40. Using sixth-order *B*-splines for case (i) little improvement occurs;

here the use of six grid lengths gives quite accurate results, although for case (ii) an improvement of the order of approximately 60 occurs.

Again it is necessary to double the number of grid lengths to maintain accuracy as the wavelength is halved, and an improvement in accuracy is evident with the higher order spline.

(d) *Current Specified on Opposite Boundaries*

Analytic solutions of Eqs. (1) and (2) for oscillations of a body of water in a channel closed at both ends ($u(0, t) = u(L, t) = 0$ for all t) are given by

$$u(x, t) = C \sin \frac{r\pi x}{L} \sin \frac{2\pi t}{T},$$
$$\xi(x, t) = \frac{TH}{2L} C \cos \frac{r\pi x}{L} \cos \frac{2\pi t}{T}, \tag{27}$$

where C is the amplitude and $T = 2L/(gH)^{1/2}$ is the period. Numerical values used were $L = 300$ km, $H = 90.8$ m, $g = 9.81$ m/sec^2, and $C = 1.0$ m/sec; the initial conditions are given by (27) with $t = 0$, a time step 1/60 of the period being employed in the integration. Two calculations were performed with (i) $r = 1.0$, (ii) $r = 2.0$, and maximum errors after 15 cycles are presented in Table VII. To check the stability of the results the calculation was continued for over 100 cycles, and results consistent with Table VII were obtained. The boundary conditions of zero flow were readily incorporated by using a knot of multiplicity $n - 1$ at the points $x = 0$ and $x = L$.

TABLE VII

Errors after 15 Cycles for Problem (d), Cases (i) and (ii)

		Fourth-order *B*-splines, number of grid lengths			Sixth-order *B*-splines, number of grid lengths		
Case		5	7	10	5	7	10
(i)	$\Delta\xi$	4.14*D*–3	7.09*D*–4	5.56*D*–5	5.26*D*–5	4.41*D*–5	4.26*D*–5
	ΔU	2.51*D*–4	6.45*D*–5	1.95*D*–5	1.38*D*–5	1.32*D*–5	1.31*D*–5
(ii)	$\Delta\xi$	2.63*D*–2	4.98*D*–3	3.30*D*–3	2.69*D*–3	1.97*D*–3	1.94*D*–3
	ΔU	9.64*D*–3	2.16*D*–3	7.20*D*–4	7.17*D*–4	6.58*D*–4	6.55*D*–4

The examples given in this section illustrate the ease with which various boundary conditions involving specification of elevations, currents, and periodic conditions can be incorporated.

Comparison of the results obtained in case (a) emphasizes the improvement in accuracy obtained using cubic spline functions rather than Hermite cubics, this improvement being comparable with that expected from the error bounds.

Results for all cases considered show that as the wavelength is halved the grid spacing must also halve to maintain accuracy (for a fixed order of spline), a result to be expected from the theoretical error estimates.

The errors computed here are maximum errors, rather than errors computed using the L_2 norm, the norm associated with the Galerkin technique, and also contain errors due to rounding, making comparison of these results with theoretical error bounds particularly difficult.

From the tables it is evident that the improvement in accuracy suggested by the error bounds, as the order of spline and number of grid intervals is increased, diminishes due to the increasing number of numerical operations required, although the error bounds give considerable insight into the variation of these errors with basis size, order of spline, and wavelength of solution.

5. Conclusions

The results presented for the different cases demonstrate how readily the various boundary conditions can be incorporated using a basis of B-splines. Using the recurrence algorithm to generate these splines allows their order to be readily changed, and the multiplicity and positions of the knots are completely flexible.

The method yields accurate results for all boundary conditions considered, the accuracy being improved in accord with theoretical error bounds as the number of basis functions and order of spline increase.

The high accuracy obtained throughout the whole spatial domain for all the problems of oceanographic interest, with a small number of grid intervals, illustrates the power of the Galerkin method when applied to these problems.

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